

Friedmann Robertson Walker models with Conformally Coupled Massive Scalar Fields are Non-integrable

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In this work we use a recently developed nonintegrability theorem of Morales and Ramis to prove that the Friedmann Robertson Walker cosmological model with a conformally coupled massive scalar field is nonintegrable.

I. INTRODUCTION

In recent years the search for nonintegrability criteria for Hamiltonian systems in the complex domain has acquired more relevance [1]-[5]. Such techniques are potentially of particular importance in cosmology because of controversies over both integrability, and the existence of chaos in cosmological models [6]-[14]. Part of the problem is that certain methods used to traditionally measure chaos in non-relativistic systems, such as the Lyapunov exponents, are no longer valid in General Relativity where there is no absolute time coordinate.

In this work we use a recently developed theorem by Morales and Ramis [1] which establishes a relation between two different concepts of integrability: the complete integrability of complex analytical Hamiltonian systems (given by Liouville's theorem) and the integrability of homogeneous linear ordinary differential equations (LODEs) in terms of Liouvillian functions in the complex plane. A Liouvillian function is a function which can be written as a combination of elementary functions, algebraic functions (solutions of polynomial equations), their indefinite integrals or exponentials of these integrals. Since we are working in the complex domain this definition includes elementary functions such as logarithms, trigonometric functions and their inverses.

The model we study in this paper is a Friedman-Robertson-Walker (FRW) cosmological model with a conformally coupled scalar field, ϕ , of mass m . We use the conformal form of the metric

$$ds^2 = a^2(\eta) \left[d\eta^2 - \frac{1}{1 - kr^2} dr^2 - r^2 d\Omega^2 \right], \quad (1)$$

with η the conformal time, $a(\eta)$ the scale factor and $k = 0, \pm 1$ the curvature.

The dynamics of this model has been discussed and studied before using numerical methods [6]-[9] but, as far as we are aware, no completely rigorous conclusion has been reached about its integrability. In [10] the integrability of a generalisation of the model studied here is considered using Painlevé analysis via the ARS algorithm [15, 16, 17]. Though there is a strong connection between integrability and the Painlevé property, and the latter has been remarkably successful in indicating possibly integral cases, it is worth noting that the lack of the Painlevé property is not a rigorous obstruction to integrability [18]. Additionally, the ARS algorithm is not a foolproof method for determining whether a system possesses the Painlevé property, and its application can lead to false conclusions [11, 12, 13], particularly when applied to determining the non-integrability of a dynamical system. Finally, certain expressions in [10] are undefined for our model, requiring a separate analysis.

The Morales-Ramis Theorem (MRT) which we use in our study rigorously provides necessary conditions for the integrability of a Hamiltonian system and so sufficient conditions for non-integrability. The theorem can be used to reduce the question of integrability to one of the existence of Liouvillian solutions of a homogeneous second-order linear ODE. This problem can in turn be solved using Kovacic's algorithm. Though complex to write down, the algorithm is, as we shall see, straightforward to apply to the problem considered here. To use the MRT for our cosmological model we first require a Hamiltonian which generates the field equations. In this case it is known [6]-[9] that a suitable Hamiltonian is

$$H = \frac{1}{2} [(p_\phi^2 + k\phi^2) - (p_a^2 + ka^2) + m^2 a^2 \phi^2] = 0, \quad (2)$$

where p_a and p_ϕ are the momenta conjugate to a and ϕ respectively.

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In the next section we give the main results of the Morales-Ramis theorem. Since the various versions of Kovacic's algorithm in the literature [1, 14, 20, 21] have slight differences in presentation and conventions, we include the version of the algorithm as used by us. We then show how the algorithm quickly determines that the Hamiltonian is nonintegrable for $k \neq 0$. Finally for the case $k = 0$ the analysis based on the invariant planes $a = p_a = 0$ and $\phi = p_\phi = 0$ is inconclusive. However, because the potential is homogeneous in this case, there exist particular nonsingular solutions which do not lie in these planes which can be used as a basis for the analysis. Fortunately the case of homogeneous potentials has been exhaustively studied by Yoshida [5] and Morales-Ramis [1] and so we can simply apply those results.

II. THE MORALES-RAMIS THEOREM

The Morales-Ramis theorem is a nonintegrability criterion: it gives a necessary condition for a Hamiltonian system to be integrable and therefore a sufficient condition for nonintegrability. The theorem is based on the analysis of the variational equations (in particular the normal variational equation, or NVE) for the perturbations of a non-equilibrium particular solution. The basic idea is that if the flow of the Hamiltonian system has a regular behaviour (is integrable), then the linearized flow along a particular integral curve given by the NVE must also be regular (integrable). Conversely if the linearized flow is nonintegrable the system as a whole will be nonintegrable.

A Hamiltonian system, X_H , of dimension n is called integrable if there exist n independent constants of the motion in involution. By considering the differential Galois group of the NVE, the theorem of Morales-Ramis links this concept of integrability to an apparently different concept of integrability – the existence of Liouvillian solutions of the NVE of X_H . The theorem may be stated as

Theorem 1 *If there are n first integrals of X_H that are independent and in involution, then the identity component of the Galois group of the NVE is abelian.*

It is known that [19] for an ODE to admit a Liouvillian solution, the identity component of its Galois group must be soluble. Hence, if the solutions are not Liouvillian, the identity component of the Galois group is not soluble and, therefore, non-Abelian.

Our strategy will therefore be:

- 1: Select a particular solution (in our case an invariant plane).
- 2: Write the variational equations and the NVE.
- 3: Check if the solutions of the NVE are Liouvillian functions.

To decide the third step, we use Kovacic's algorithm [20] which we now turn to describe.

III. KOVACIC'S ALGORITHM

Kovacic's algorithm provides a procedure for computing the Liouvillian solutions of a homogeneous linear second order differential equation. If the algorithm terminates negatively, we can conclude that no such solutions exist.

Let $\mathbb{C}(x)$ be the field of rational complex functions (ratios of polynomials in x with complex coefficients). It is well-known that by using the change of dependent variable

$$y = \xi \exp\left(\frac{1}{2} \int b \, dx\right) \quad (3)$$

the second order homogeneous LODE

$$y'' + b(x)y' + c(x)y = 0$$

can be transformed to the so-called reduced invariant form

$$\xi'' - g\xi = 0, \quad (4)$$

where

$$g(x) = \frac{1}{2}b'(x) + \frac{1}{4}b(x)^2 - c(x). \quad (5)$$

Note that, if $b(x)$ and $c(x) \in \mathbb{C}(x)$ then $g(x) \in \mathbb{C}(x)$.

Moreover, using a further change of variables $v = \xi'/\xi$, equation (4) is transformed into the Riccati equation

$$v' + v^2 = g. \quad (6)$$

Now equation (4) is integrable, if and only if equation (6) has an algebraic solution, that is v solves a polynomial equation $f(v) = 0$, where the degree of f (the minimal polynomial) in v belongs to the set $L = \{1, 2, 4, 6, 12\}$.

Kovacic's algorithm can be divided into three main steps: the first step is the determination of the subset of L relevant for the LODE under consideration; the two other steps are devoted respectively to determining the existence of the minimal polynomial, and its construction. If the algorithm does not terminate successfully (ie, equation (6) has no algebraic solution) then equation (4) has no solution in terms of Liouvillian functions.

In the version used of the algorithm we essentially follow [1, 14, 21, 22]. Let

$$g = g(x) = \frac{s(x)}{t(x)}, \quad (7)$$

with $s(x), t(x)$ relatively prime polynomials, and $t(x)$ monic. Define the function h on the set $L_{max} = \{1, 2, 4, 6, 12\}$ by $h(1) = 1$, $h(2) = 4$, $h(4) = h(6) = h(12) = 12$.

Step 1 (determination of possible orders of the minimal polynomial)

If $t(x) = 1$ then set $m = 0$, else factorize $t(x)$ into monic relatively prime polynomials

$$t(x) = t_1(x) t_2^2(x) \dots t_m^m(x),$$

where t_i have no multiple roots and $t_m \neq 1$.

Then

1.1 Let Γ' be the set of roots of $t(x)$ (i.e., the singular points in the finite complex plane) and let $\Gamma = \Gamma' \cup \infty$ be the set of singular points.

Then the order of a singular point $c \in \Gamma'$ is, as usual, $o(c) = i$ if c is a root of multiplicity i of t_i . The order at infinity is defined by $o(\infty) = \max(0, 4 + \deg(s) - \deg(t))$. We call $m^+ = \max(m, o(\infty))$.

For $0 \leq i \leq m^+$, denote by $\Gamma_i = \{c \in \Gamma \mid o(c) = i\}$ the subset of all elements of order i .

1.2 If $m^+ \geq 2$ then we write $\gamma_2 = \text{card}(\Gamma_2)$, else $\gamma_2 = 0$. Then we compute

$$\gamma = \gamma_2 + \text{card} \left(\bigcup_{\substack{3 \leq k \leq m^+ \\ k \text{ odd}}} \Gamma_k \right).$$

1.3 For the singular points of order one or two, $c \in \Gamma_2 \cup \Gamma_1$, we compute the principal parts of g :

$$g_c = \alpha_c(x-c)^{-2} + \beta_c(x-c)^{-1} + O(1),$$

if $c \in \Gamma'$, and

$$g_\infty = \alpha_\infty x^{-2} + \beta_\infty x^{-3} + O(x^{-4}),$$

for the point at infinity.

1.4 We define the subset L' (of all possible values for the degree of minimal polynomial) as $\{1\} \subset L'$ if $\gamma = \gamma_2$, $\{2\} \subset L'$ if $\gamma \geq 2$ and $\{4, 6, 12\} \subset L'$ if $m^+ \leq 2$.

1.5 We have the three following mutually exclusive cases:

1.5.1 If $m^+ > 2$, then $L = L'$.

1.5.2 Define $\Delta_c = \sqrt{1 + 4\alpha_c}$. If $m^+ \leq 2$ and $\forall c \in \Gamma_1 \cup \Gamma_2$, $\Delta_c \in \mathbb{Q}$, then $L = L'$.

1.5.3 If cases (1.5.1) and (1.5.2) do not hold, then $L = L' - \{4, 6, 12\}$.

1.6 If $L = \emptyset$, then equation (4) is non-integrable with Galois group $SL(2, \mathbb{C})$, else one writes n for the minimum value in L .

For the second and third steps of the algorithm we consider a fixed value of n .

Step 2

2.1 If ∞ has order 0 we write the set

$$E_\infty = \left\{ 0, \frac{h(n)}{n}, 2\frac{h(n)}{n}, 3\frac{h(n)}{n}, \dots, n\frac{h(n)}{n} \right\}.$$

2.2 If c has order 1, then $E_c = \{h(n)\}$.

2.3 If $n = 1$, for each c of order 2 we define

$$E_c = \left\{ \frac{1}{2}(1 + \Delta_c), \frac{1}{2}(1 - \Delta_c) \right\}$$

2.4 If $n \geq 2$, for each c of order 2, we define

$$E_c = \mathbb{Z} \cap \left\{ \frac{h(n)}{2}(1 - \Delta_c) + \frac{h(n)}{n}k\Delta_c : k = 0, 1, \dots, n \right\}.$$

2.5 If $n = 1$, for each singular point of even order 2ν , with $\nu > 1$, we compute the numbers α_c and β_c defined (up to a sign) by the following conditions:

2.5.1 If $c \in \Gamma'$,

$$g_c = \left\{ \frac{\alpha_c}{(x-c)^\nu} + \sum_{i=2}^{\nu-1} \frac{\mu_{i,c}}{(x-c)^i} \right\}^2 + \frac{\beta_c}{(x-c)^{\nu+1}} + O(x-c)^{-\nu}$$

and we write

$$\sqrt{g_c} := \alpha_c(x-c)^{-\nu} + \sum_{i=2}^{\nu-1} \mu_{i,c}(x-c)^{-i}.$$

2.5.2 If $c = \infty$,

$$g_\infty = \left\{ \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i \right\}^2 - \beta_\infty x^{\nu-3} + O(x^{\nu-4}),$$

and we write

$$\sqrt{g_\infty} := \alpha_\infty x^{\nu-2} + \sum_{i=0}^{\nu-3} \mu_{i,\infty} x^i.$$

Then for each c as above, we compute

$$E_c = \left\{ \frac{1}{2} \left(\nu + \epsilon \frac{\beta_c}{\alpha_c} \right) : \epsilon = \pm 1 \right\},$$

and the sign function on E_c is defined by

$$\text{sign} \left(\frac{1}{2} \left(\nu + \epsilon \frac{\beta_c}{\alpha_c} \right) \right) = \epsilon,$$

being +1 if $\beta_c = 0$.

2.6 If $n = 2$, for each c of order ν , with $\nu \geq 3$, we write $E_c = \{\nu\}$.

Step 3

3.1 For n fixed, we try to obtain elements $\mathbf{e} = (e_c)_{c \in \Gamma}$ in the Cartesian product $\prod_{c \in \Gamma} E_c$, such that:

(i) $d(\mathbf{e}) := n - \frac{n}{h(n)} \sum_{c \in \Gamma} e_c$ is a non-negative integer,

(ii) If $n = 2$ or $n = 6$ then \mathbf{e} has an even number of elements which are odd integers

(iii) when $n = 4$, then \mathbf{e} has at least two elements not divisible by 3, and the sum of all elements not divisible by 3 is divisible by 3.

If no such set \mathbf{e} is obtained, we select the next value in L and repeat Step 2, else n is the maximum value in L and the Galois group is $SL(2, \mathbb{C})$ (and equation (4) is non-integrable).

3.2 For each family \mathbf{e} as above, we try to obtain a rational function Q and a polynomial P , such that

$$(i) \quad Q = \frac{n}{h(n)} \sum_{c \in \Gamma'} \frac{e_c}{x-c} + \delta_{n1} \sum_{c \in \bigcup_{\nu > 1} \Gamma_{2\nu}} \text{sign}(e_c) \sqrt{g_c},$$

where δ_{n1} is the Kronecker delta.

(ii) P is a polynomial of degree $d(\mathbf{e})$ and its coefficients are found as a solution of the (in general, overdetermined) system of equations

$$\begin{aligned} P_{-1} &= 0, \\ P_{i-1} &= -(P_i)' - QP_i - (n-i)(i+1)gP_{i+1}, \quad n \geq i \geq 0, \\ P_n &= -P. \end{aligned}$$

If a pair (P, Q) as above is found, then equation (4) is integrable and the Riccati equation (6) has an algebraic solution v given by any root v of the equation

$$f(v) = \sum_{i=0}^n \frac{P_i}{(n-i)!} v^i = 0.$$

If no pair as above is found we take the next value in L and we go to Step 2. If n is the greatest value in L then the Galois group of (4) is $SL(2, \mathbb{C})$ and the ODE is non-integrable.

IV. APPLICATION AND RESULT

A. The case $k \neq 0$

We apply the theorem of Morales-Ramis to (2). We choose as our set of non-equilibrium particular solutions the invariant plane $p_a = a = 0$. The NVEs relative to this plane are

$$\frac{d^2 \delta a}{dt^2} = (-k + m^2 \phi^2) \delta a. \quad (8)$$

Changing the independent variable to ϕ and renaming $\delta a = y$, we obtain the equation

$$\frac{d^2 y}{d\phi^2} + \frac{1}{\phi} \frac{dy}{d\phi} + \left(\frac{m^2}{k} - \frac{1}{\phi^2} \right) y = 0. \quad (9)$$

This equation is a second-order, linear and homogeneous ODE with coefficients which are rational functions of ϕ , and we can therefore apply Kovacic's algorithm to determine any Liouvillian solutions.

Using (3) and (5) we transform (9) into the reduced invariant form (4)

$$\xi'' = \left(\frac{3k - 4m^2 \phi^2}{4k\phi^2} \right) \xi. \quad (10)$$

Lemma 1 Equation (9) has no Liouvillian solutions when $k \neq 0$ and $m \neq 0$.

Proof: by application of Kovacic's algorithm to equation (10).

Step 1

$g(\phi)$ has one finite pole, at $\phi = 0$, of order 2 and the pole at infinity, of order 4 (since, by assumption, we are treating the massive case, $m \neq 0$). This implies that $m^+ = 4$ and $\gamma = \gamma_2 = 1$. Since the pole at $\phi = 0$ belongs to Γ_2 , we calculate the Laurent series (when $k \neq 0$) as

$$g_0 = \frac{3}{4} \phi^{-2} - \frac{m^2}{k}$$

Hence $\alpha_0 = \frac{3}{4}$ and $\beta_0 = 0$. Thus we have $L = \{1\}$.

Step 2

Because $L = \{1\}$ the unique value for n is $n = 1$. Through the items 2.3 and 2.5 we calculate the sets E_c . From 2.3 we have that $E_0 = \left\{ \frac{3}{2}, -\frac{1}{2} \right\}$. In item 2.5.2 we need to expand g around $\phi = \infty$. Doing this we obtain

$$g_\infty = \frac{3}{4} \phi^{-2} - \frac{m^2}{k} \implies E_\infty = \{1\}.$$

Summarizing,

$$E_0 = \left\{ \frac{3}{2}, -\frac{1}{2} \right\} \quad \text{and} \quad E_\infty = \{1\}.$$

Step 3

In this step we need to calculate $\prod_{c \in \Gamma} E_c$, using the sets determined in the previous step. We obtain the set of sets given by

$$\prod_{c \in \Gamma} E_c = \left\{ \left\{ \frac{3}{2}, 1 \right\}, \left\{ -\frac{1}{2}, 1 \right\} \right\}.$$

From 3.1(i) we calculate the values of $d(\mathbf{e})$ as $d = -\frac{3}{2}$ and $d = \frac{1}{2}$ respectively. Since neither of these values satisfies 3.1(i) and there are no other values of n in L , the Galois group of (10) is $SL(2, \mathbb{C})$, equation (10) is nonintegrable in terms of Liouvillian functions, and therefore the system represented by the Hamiltonian (2) is also nonintegrable when $k \neq 0$. This completes the proof.

B. The case $k = 0$

Lemma 2 When $k = 0$ the only first integral of the Hamiltonian system (2) is the Hamiltonian, and the system is therefore nonintegrable.

Proof: In order to prove this lemma we observe that when $k = 0$ the Hamiltonian (2) has a homogeneous potential. Hamiltonians with homogeneous potentials have been exhaustively studied using the MRT and particular results obtained which we now outline.

Let

$$H = \frac{1}{2} \sum_{i=1}^n p_i^2 + V(q_1, \dots, q_n) \quad (11)$$

where A is a constant and V is a homogeneous potential, i.e. $V(A\vec{Q}) = A^g V(\vec{Q})$ with g being the degree of the potential. To put our Hamiltonian in the form (11) we perform the canonical transformation, $x = ia$ and $P_x = -iP_a$, after which

$$H = \frac{1}{2} [p_x^2 + p_\phi^2 - m^2 x^2 \phi^2] \quad (12)$$

and $g = 4$. The MRT for homogeneous potentials is given by [1, 5].

Theorem 2 Let $V(q_1, \dots, q_n)$ be a homogeneous potential function of integer degree g , c a solution of the equation $c = \vec{V}'(c)$, and λ_i (the Yoshida coefficients) the eigenvalues of the matrix $V''(c)$. One of these eigenvalues is trivial, in that it corresponds to the tangential variational equation, and has value $g - 1$.

If a Hamiltonian system of the form (11) is completely integrable (with holomorphic or meromorphic first integrals) then each pair (g, λ_i) belongs to one of the following list (where we do not consider the trivial case $g = 0$)

- (1) $(g, p + p(p - 1)g/2)$
- (2) $(2, \text{arbitrary complex number})$
- (3) $(-2, \text{arbitrary complex number})$
- (4) $(-5, \frac{49}{40} - \frac{1}{40}(\frac{10}{3} + 10p)^2)$
- (5) $(-5, \frac{49}{40} - \frac{1}{40}(4 + 10p)^2)$
- (6) $(-4, \frac{9}{8} - \frac{1}{8}(\frac{4}{3} + 4p)^2)$
- (7) $(-3, \frac{25}{24} - \frac{1}{24}(2 + 6p)^2)$
- (8) $(-3, \frac{25}{24} - \frac{1}{24}(\frac{3}{2} + 6p)^2)$
- (9) $(-3, \frac{25}{24} - \frac{1}{24}(\frac{6}{5} + 6p)^2)$
- (10) $(-3, \frac{25}{24} - \frac{1}{24}(\frac{12}{5} + 6p)^2)$
- (11) $(3, -\frac{1}{24} + \frac{1}{24}(2 + 6p)^2)$
- (12) $(3, -\frac{1}{24} + \frac{1}{24}(\frac{3}{2} + 6p)^2)$
- (13) $(3, -\frac{1}{24} + \frac{1}{24}(\frac{6}{5} + 6p)^2)$
- (14) $(3, -\frac{1}{24} + \frac{1}{24}(\frac{12}{5} + 6p)^2)$
- (15) $(4, -\frac{1}{8} + \frac{1}{8}(\frac{4}{3} + 4p)^2)$
- (16) $(5, -\frac{49}{40} + \frac{1}{40}(\frac{10}{3} + 10p)^2)$
- (17) $(5, -\frac{49}{40} + \frac{1}{40}(4 + 10p)^2)$
- (18) $(g, \frac{1}{2}(\frac{g-1}{g} + p(p+1)g))$

where p is an arbitrary integer.

For the system represented by (12) the only non-trivial

Yoshida coefficient is $\lambda = -1$. For $g = 4$ the only possibilities for satisfying Theorem 3 are (1), (15) and (18). For all these cases there are no integer values of p which solve $\lambda = -1$, and we conclude that the system represented by (12) is nonintegrable. This completes the proof of lemma 2.

We can now enunciate the following theorem

Theorem 3 The Friedmann Robertson Walker model with a conformally coupled massive scalar field represented by the Hamiltonian (2) is not completely integrable.

Proof: By lemma 1 there are no Liouvillian solutions of the NVE for the plane $a = p_a = 0$. This implies that the identity component of its Galois group is not soluble and therefore non-Abelian. Using theorem 1 we have that the only first integral of the Hamiltonian system is the Hamiltonian itself, and that the system is not completely integrable in this case. By lemma 2 the Hamiltonian system is also not completely integrable when $k = 0$. Therefore the Hamiltonian system represented by (2) is nonintegrable for all values of k .

V. CONCLUSION

From our analysis we have shown rigorously using analytic methods that FRW universes with a conformally coupled massive scalar field are nonintegrable. This is compatible with results from numerical analysis based on Poincaré sections [6] which indicate that the behaviour of the system is mathematically chaotic.

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